DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

265. Proposed by G. W. GREENWOOD, M. A., Dunbar, Pa.

Obtain the reduced cubic $4\theta^3-I\theta+J=0$ of the biquadratic $ax^4+4bx^3+6cx^2+4dx+e=0$.

I. Solution by the PROPOSER.

Assume $ax^4+4bx^3+6cx^2+4dx+e\equiv a(x^2+2mx+p)(x^2+2nx+q)$.

$\therefore a(m+n)=2b$; $a(4mn+p+q)=6c$; $a(mq+np)=2d$; $apq=e$.

Let $amn=c-\theta$; then $a(p+q)=2c+4\theta$.

Substituting in the identity

$$\begin{vmatrix} 1 & 1 & 0 \\ m & n & 0 \\ p & q & 0 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 0 \\ n & m & 0 \\ q & p & 0 \end{vmatrix} \equiv \begin{vmatrix} 2 & m+n & p+q \\ m+n & 2mn & mq+np \\ p+q & mq+np & 2pq \end{vmatrix} \equiv 0$$

we obtain $\dfrac{8}{a^3}\begin{vmatrix} a & b & c+2\theta \\ b & c-\theta & d \\ c+2\theta & d & e \end{vmatrix}=0$; *i. e.*, $4\theta^3-I\theta+J=0$.

II. Solution by L. E. NEWCOMB, Los Gatos, Cal.

$$ax^4+4bx^3+6cx^2+4dx+e\equiv x^4+\frac{4bx^3}{a}+\frac{6cx^2}{a}+\frac{4dx}{a}+\frac{e}{a}=0\ldots(1).$$

Let $x=y/a$; then (1) becomes $y^4+4by^3+6acy^2+4a^2dy+a^3e=0\ldots(2)$.

$y^4+4by^3+6acy^2+4a^2dy+a^3e+(\alpha y+\beta)^2=(y^2+2by+\lambda)^2$, by Ferrari's solution; whence, by equating coefficients of like powers of y, and eliminating α and β, $\lambda^3-3ac\lambda^2+a^4(4bd-ae)\lambda-a^3(2d^2+2b^2e-3ace)=0$. Let $4^{\frac{1}{3}}\theta+ae=\lambda$, then

$$4\theta^3-4^{\frac{1}{3}}a^2(3c^2+ac-4bd)\theta+a^3[c(4bd-ac-2c^2)-(2cd^2+2b^2e-3ace)]=0\ldots(3),$$

which is of the required form.

266. Proposed by L. E. NEWCOMB, Los Gatos, Calif.

Find the nth term and the sum of n terms of the series $1+3+7+17+\ldots$.

I. Solution by THEODORE L. DeLAND, Treasury Department, Washington, D. C.

Let u_1, u_2, u_3, u_4, ..., u_n be the terms of the series, in the Calculus of Finite Differences, and let Δu_1, $\Delta^2 u_1$, $\Delta^3 u_1$, and $\Delta^4 u_1$ be the symbols for the different orders of differences.

The differences and the terms of this special series may be arranged as follows:

$$
\begin{array}{llllllllll}
u_1 = & 1 & 3 & 7 & 17 & . & . & . & & \\
\Delta u_1 = & . & 2 & 4 & 10 & . & . & . & . & \\
\Delta^2 u_1 = & . & . & 2 & 6 & . & . & . & . & . \\
\Delta^3 u_1 = & . & . & . & 4 & . & . & . & . & .
\end{array}
$$

This shows that the third difference of u_1 or $\Delta^3 u_1$, is constant; and that, therefore, the fourth, and all higher differences, in this series, must vanish.

We have always, $u_1 = u_1$; and then we have for the next term, $u_2 = u_1 + \Delta u_1 = u_1(1+\Delta)$; and then, $u_3 = u_1 + \Delta u_1 + \Delta(u_1 + \Delta u_1) = u_1 + \Delta u_1 + \Delta u_1 + \Delta^2 u_1 = u_1 + 2\Delta u_1 + \Delta^2 u_1 = u_1(1+\Delta)^2$; and so on for higher orders; and in which Δ may be considered, first, as a symbol of operation, and second, as a symbol of quantity.

The symbols and operations may now be exhibited as follows, as they conform to the law of the Binomial Theorem:

$$
\begin{aligned}
&u_1 = u_1 \\
&u_2 = u_1(1+\Delta) \\
&u_3 = u_1(1+\Delta)^2 \\
&u_4 = u_1(1+\Delta)^3 \\
&\ldots\ldots\ldots \\
&u_n = u_1(1+\Delta)^{n-1}
\end{aligned}
$$

Expand the term for u_n and we have for its value:

$$u_n = u_1[1 + (n-1)\Delta + \frac{(n-1)(n-2)}{2}\Delta^2 + \frac{(n-1)(n-2)(n-3)}{2\times 3}\Delta^3 + \ldots +].(A).$$

Let S_n = the value of the sum of n terms and we have:

$$S_n = u_1 + u_2 + u_3 + u_4 + \ldots + u_n.$$

We also have for the second member:

$$S_n = u_1[1 + (1+\Delta) + (1+\Delta)^2 + (1+\Delta)^3 + (1+\Delta)^4 + \ldots + (1+\Delta)^{n-1}]\ldots(B).$$

Sum (B) and we have:

$$S_n = \frac{u_1[(1+\Delta)^n - 1]}{\Delta}.$$

Expand the term, $(1+\Delta)^n$, subtract 1, and divide by Δ, and we have:

$$S_n = u_1[n + \frac{n(n-1)}{2}\Delta + \frac{n(n-1)(n-2)}{3!}\Delta^2 + \frac{n(n-1)(n-2)(n-3)}{4!}\Delta^3 + \ldots]\ldots(C).$$

In (A) and (C) remove the brackets so as to unite the symbols of operation and the symbols of quantity and we have:

$$u_n=u_1+(n-1)\triangle u_1+\frac{(n-1)(n-2)}{2!}\triangle^2 u_1+\frac{(n-1)(n-2)(n-3)}{3!}\triangle^3 u_1+\ldots(D);$$

and

$$S_n=nu_1+\frac{n(n-1)}{2}\triangle u_1+\frac{n(n-1)(n-2)}{3!}\triangle^2 u_1$$
$$+\frac{n(n-1)(n-2)(n-3)}{4!}\triangle^3 u_1+\ldots(E).$$

In (D) and (E) substitute the values, $u_1=1$, $\triangle u_1=2$, $\triangle^2 u_1=2$, and $\triangle^3 u_1=4$, from the problem and its differences, and we have, after reduction:

$$u_n=\tfrac{1}{3}[2n^3-9n^2+19n-9]\ldots(F);\text{ and}$$
$$S_n=\tfrac{1}{6}[n^4-4n^3+11n^2-2n]\ldots(G).$$

Equations (F) and (G) are true for all values of n for the special series under consideration. When $n=4$, $u_n=u_4=17$, and $S_n=S_4=28$, as may be seen by inspecting the series in the problem.

But equations (D) and (E) are perfectly general when the series follows any regular law of progression; as we have to know, only, the value of the leading term, and the leading differences up to the difference that vanishes, to find the value of any term in a series and the sum of that series.

II. Solution by L. E. NEWCOMB, Los Gatos, Cal., and G. W. GREENWOOD, M. A., Dunbar, Pa.

Let $S\equiv u_1+u_2x+u_3x^2+\ldots$ where $u_1=1$, $u_2=3$, $u_3=7$, and, in general, $u_n=2u_{n-1}+u_{n-2}$, x, of course, being less than unity, numerically.

$\therefore(1-2x-x^2)S=u_1+(u_2-2u_1)x$; *i. e.*,

$$S=\frac{1+x}{1-2x-x^2}=\frac{A}{1-\alpha x}+\frac{B}{1-\beta x}$$

where $\alpha=1+\sqrt{2}$, $\beta=1-\sqrt{2}$, $A=\frac{1}{2}(1+\sqrt{2})$, $B=\frac{1}{2}(1-\sqrt{2})$.

$\therefore u_n=A\alpha^{n-1}+B\beta^{n-1}=\frac{1}{2}[(1+\sqrt{2})^n+(1-\sqrt{2})^n]$.

Let $S_n=u_1+u_2+\ldots+u_n$; $S_n(1-2-1)=u_1+u_2-2u_1-3u_n-u_{n-1}$; *i. e.*, $S_n=\frac{1}{2}[3u_n+u_{n-1}-2]$.

Solved in a similar manner by J. Scheffer.

CALCULUS.

219. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Evaluate (a) $\int_0^{\frac{1}{2}\pi}\frac{\sin mx\sin nx}{\sin x}dx$; (b) $\int_0^{\frac{1}{2}\pi}\frac{\cos mx\sin nx}{\sin x}dx$, where n is a positive integer. Also, modify the result for the case of m an integer.